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The discrete first, second and thirty-fourth Painlevé hierarchies

Clio Cresswell† and Nalini Joshi
Department of Pure Mathematics, University of Adelaide, Adelaide SA 5005, Australia

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Abstract. The discrete first and second Painlevé equations (dP_I and dP_{II}) are integrable difference equations which have classical first, second or third Painlevé equations (P_I, P_{II} or P_{III}) as continuum limits. dP_I and dP_{II} are believed to be integrable because they are discrete isomonodromy conditions for associated (single-valued) linear problems. An infinite hierarchy of integrable difference equations that share the same linear deformation problem as dP_I was shown to exist by Cresswell and Joshi. In this paper, we recall the results shown for dP_I and show how to deduce a hierarchy for dP_{II}. Each member of the respective hierarchies is shown to be generated by difference recursion operators. Furthermore, we show that continuum limits of these difference hierarchies lead to the P_I, P_{II} and P_{III} hierarchies. Finally, we construct Miura transformations of the dP_{II} hierarchy and show that these lead to the hierarchy of the discrete thirty-fourth Painlevé equation.

1. Introduction

The first discrete Painlevé equation (dP_I)

$$x_{n+1} + x_n + x_{n-1} = \frac{C_1 + C_2 n}{x_n} + C_3 \tag{1.1}$$

and the second discrete Painlevé equation (dP_{II})

$$x_{n+1} + x_{n-1} = \frac{(C_1 + C_2 n)x_n + C_3}{1 - x_n^2} \tag{1.2}$$

have been the subject of many investigations [1, 2] because of the remarkable properties they share with their continuum limits. These are the well known integrable nonlinear classical ordinary differential equations (ODEs) called the Painlevé equations. In this paper, we add to their remarkable features by showing that each lies at the base of an infinite hierarchy of nonlinear difference equations that are generated by a recursion operator.

The scaled continuum limit $x_n = 1 + h^2 u(t)$, $t = nh$, with $C_1 = -3 + r_1 h^2$, $C_2 = r_2 h^5$, $C_3 = 3 - \beta - r_3 h^4$, where the r_i ($i = 1, 2, 3$) are constants, applied to dP_I yields a scaled and translated version of the classical P_I

$$u'' = 6u^2 + t$$

as $h \to 0$. The scaled continuum limit $x_n = hu(t)$, $t = nh$, with $C_1 = 2 + r_1 h^3$, $C_2 = r_2 h^3$, $C_3 = r_3 h^3$, where the r_i ($i = 1, 2, 3$) are constants, applied to dP_{II} yields a scaled and translated version of the classical P_{II}

$$u'' = 2u^3 + tu + \alpha$$

† Permanent address: School of Mathematics, University of New South Wales, Sydney NSW 2052, Australia.

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as $h \to 0$ (α is a constant). It is well known that $\mathrm{dP_{II}}$ also has a continuum limit to the third Painlevé equation [3] ($\mathrm{P_{III}}$)

$$u''(t) = \frac{(u')^2}{u} - \frac{u'}{t} + \frac{\alpha u^2 + \beta}{t} + \gamma u^3 + \frac{\delta}{u}$$

where α, β, γ, δ are constants.

A characteristic feature of the classical Painlevé equations is that they possess the Painlevé property, i.e. all movable singularities of all solutions are poles. Painlevé [4], Gambier [5] and Fuchs [6] identified six second-order nonlinear ODEs (under some mild conditions) as the only ones with the Painlevé property whose general solutions are new transcendental functions. $\mathrm{P_I}$, $\mathrm{P_{II}}$ and $\mathrm{P_{III}}$ were the first three of these six equations. See the list of 50 canonical classes identified by the Painlevé school in Ince [7]. $\mathrm{P_{XXXIV}}$ referred to below is the thirty-fourth member of this list.

The Painlevé equations and $\mathrm{dP_I}$, $\mathrm{dP_{II}}$ play important roles in mathematics and physics. The generic solutions of the Painlevé equations are higher transcendental functions that cannot be expressed in terms of the classical special functions [7]. The Painlevé equations are integrable reductions of soliton equations [8, 9]. Their discrete versions appear in important physical applications such as two-dimensional quantum gravity [1, 10, 11].

The Painlevé equations are considered integrable because they are isomonodromy conditions for associated (single-valued) linear systems of differential equations [12, 13]. There is strong evidence that such integrability is related to the Painlevé property [14–16].

A discrete version of the Painlevé property, called the singularity confinement property, was proposed by Grammaticos, Ramani *et al* [2, 17] who used it to derive discrete versions of the Painlevé equations. Although it is now known that this property alone is not sufficient for integrability [18], the discrete Painlevé equations found by Grammaticos, Ramani *et al* are known to be integrable in the sense that they are compatibility conditions that ensure isomonodromy for associated linear problems [19, 20]. The confinement property is equivalent to the well-posedness of the discrete equation for x_n, even through apparent singularities of the equation, on the complex sphere of x-values in both forward and backward evolution in n. For example, in the case of $\mathrm{dP_{II}}$ (equation (1.2)), if $x_{n-1} = b$, $x_n = \pm 1$ (a singularity of the map), then $x_{n+1} = \infty$, $x_{n+2} = \mp 1$ and x_{n+3} is an analytic function of b, ensuring that the map remains well-posed through a singularity [17].

There exist many other properties of the Painlevé equations that are shared by their discrete counterparts. Both can be rewritten in terms of so-called τ functions that have no movable singularities [21, 22], both possess so-called Miura and Bäcklund transformations that map the equation to another integrable equation (or to the same one with different parameters) [23–26]. $\mathrm{P_{II}}$ is well known to be related by a Miura transformation to $\mathrm{P_{XXXIV}}$ [7].

In this paper we describe further similarities between the continuous and discrete settings by constructing integrable hierarchies of equations associated with the same linear problem.

Integrable hierarchies of partial differential equations (PDEs) and ODEs as well as the links between them are well known. For example, the modified Korteweg–de Vries (mKdV) hierarchy (using $\partial = \frac{\partial}{\partial x}$)

$$W_{t_{2i+1}} + \partial(\partial + 2W)L_i[W_x - W^2] = 0 \qquad i = 1, 2, 3, \ldots$$

where

$$L_{i+1,x} := (\partial^3 + 4U\partial + 2U_x)L_i \qquad L_1[U] := U$$

may be obtained from the Korteweg–de Vries hierarchy [27]

$$U_{t_{2i+1}} + \partial L_{i+1}[U] = 0 \qquad i = 1, 2, 3, \ldots$$

via the Miura transformation $U = W_x - W^2$ [28]. In turn, the $\mathrm{P_{II}}$ hierarchy

$$\left(\frac{\mathrm{d}}{\mathrm{d}X} + 2V\right)L_i[V' - V^2] - VX - \alpha_i = 0 \qquad i = 1, 2, 3, \ldots \tag{1.3}$$

may be found through a similarity reduction of the mKdV hierarchy [29]

$$W = \frac{1}{[(2i+1)t_{2i+1}]^{1/(2i+1)}} V(X) \qquad X = \frac{x}{[(2i+1)t_{2i+1}]^{1/(2i+1)}}.$$

Also, the $\mathrm{P_{XXXIV}}$ hierarchy

$$\begin{aligned}[2L_i[Y] - X]\frac{\mathrm{d}^2}{\mathrm{d}X^2}(L_i[Y]) - \left[\frac{\mathrm{d}}{\mathrm{d}X}(L_i[Y])\right]^2 + \frac{\mathrm{d}}{\mathrm{d}X}(L_i[Y])\\ +[2L_i[Y] - X]^2 Y - \alpha_i(1-\alpha_i) = 0\end{aligned}$$

may be constructed from the $\mathrm{P_{II}}$ hierarchy through the Miura transformation [30]

$$V = -\left[\frac{\mathrm{d}}{\mathrm{d}X}(L_i[Y]) - \alpha_i\right] / [2L_i[Y] - X].$$

Equation (1.3) expresses an infinite hierarchy of equations through the recursive action of one operator L_i. Such an expression also allows us to deduce Miura and Bäcklund transformations for the whole hierarchy (as above) and corresponding special integrals for the whole hierarchy [31]. (In the case of PDEs, it is well known that the existence of such operators is related to Lie algebras underlying soliton equations [32–34].) Here we are interested in deriving a corresponding expression for the discrete setting.

The paper is organized as follows. In section 2, we recall the isomonodromy problem for $\mathrm{dP_I}$ and extend on the work presented in [35] by deriving the associated infinite hierarchy of integrable equations in terms of difference operators. We provide explicit examples up to sixth order and continuum limits of the second- and fourth-order equations. This is repeated for $\mathrm{dP_{II}}$ in section 3. In section 4, we construct the $\mathrm{dP_{XXXIV}}$ hierarchy by applying Miura transformations to equations in the $\mathrm{dP_{II}}$ hierarchy. We provide explicit examples up to fourth order. In the following, all occurrences of c_i, r_i denote constants.

2. The $\mathrm{dP_I}$ hierarchy

In this section, we derive the $\mathrm{dP_I}$ hierarchy in operator form and give the first few members of the hierarchy explicitly. We also show that their continuum limits are members of both the $\mathrm{P_I}$ and $\mathrm{P_{II}}$ hierarchies found in [29, 35, 36].

2.1. The hierarchy

The linear problem associated with $\mathrm{dP_I}$ [19] is

$$x_n\phi_{n+1} = \lambda\phi_n - \phi_{n-1} \tag{2.1}$$

$$\frac{\partial\phi_n}{\partial\lambda} = a_n\phi_{n+1} + b_n\phi_n \tag{2.2}$$

where ϕ, x, a, b depend on a discrete variable n; ϕ, a, b depend on the continuous variable λ, and a, b are rational in λ.

The compatibility conditions of equations (2.1) and (2.2) are

$$b_{n+1} - b_{n-1} + \lambda\left(\frac{a_{n+1}}{x_{n+1}} - \frac{a_n}{x_n}\right) = 0 \tag{2.3}$$

$$\frac{\lambda^2}{x_n}\left(\frac{a_{n+1}}{x_{n+1}} - \frac{a_n}{x_n}\right) + \frac{\lambda}{x_n}(b_{n+1} - b_n) - \frac{1}{x_n}\left(x_n\frac{a_{n+1}}{x_{n+1}} - a_{n-1} + 1\right) = 0. \tag{2.4}$$

Define

$$p_n := \frac{a_n}{x_n} \qquad \text{and} \qquad q_n := b_n + b_{n-1}.$$

Then b can be eliminated from equation (2.4) by using (2.3). The result is

$$-\lambda^2(p_{n+1} - p_n) + x_{n+1}p_{n+2} + x_n p_{n+1} - x_n p_n - x_{n-1}p_{n-1} + 2 = 0. \tag{2.5}$$

We concentrate on results for p_n. Corresponding results for q_n can be obtained from equation (2.3).

Equation (2.5) suggests that p_n is polynomial in λ but gives no restriction on the degree of the polynomial. (More generally it could be rational in λ, but for simplicity we restrict our analysis to the polynomial case.) Below, we generate a sequence of solutions p_n of equation (2.5) where the sequence is indexed by the degree.

First, to capture the entire sequence in closed form rewrite (2.5) as

$$-\lambda^2(\Delta^2 p_n + \Delta p_n) + 3\Delta(x_n\Delta p_n) + 4\Delta(x_n p_n) - 2(\Delta x_n)\cdot(\Delta p_n + p_n)$$
$$+\Delta^2 x_n(\Delta p_n) + \Delta^2(x_n p_n) + 2 = 0$$

where $\Delta p_n := p_{n+1} - p_n$, or

$$[\lambda^2(\Delta^2 + \Delta) - \mathcal{J}]p_n = 2 \tag{2.6}$$

where

$$\mathcal{J} := 3\Delta(x_n\Delta) + 4\Delta(x_n) - 2(\Delta x_n)\cdot(\Delta + 1) + \Delta^2 x_n(\Delta) + \Delta^2(x_n).$$

Notice that $(\Delta^2 + \Delta)p_n = p_{n+2} - p_{n+1}$.

Now take p_n to be polynomial in λ

$$p_n := \sum_{k=0}^{l} P_{k,n}\lambda^k.$$

Substituting this expression into (2.6) and solving for the coefficients $P_{k,n}$ we obtain

$$P_{l,n} = c_l \qquad P_{l-1,n} = c_{l-1}$$
$$P_{k-2,n} = (\Delta^2 + \Delta)^{-1}\mathcal{J}P_{k,n} \qquad \text{for} \quad 2 \leqslant k \leqslant l$$

where $(\Delta^2 + \Delta)^{-1}F_n := \sum_{k=0}^{n-2} F_k$ is the discrete analogue of the nonlocal operator $\partial^{-1}F = \int^x F\,\mathrm{d}x$. These lead to the compatibility conditions

$$\mathcal{J}P_{1,n} = 0 \qquad \text{and} \qquad \mathcal{J}P_{0,n} + 2 = 0.$$

Note that the equation for $P_{1,n}$ is homogeneous, whereas the equation for $P_{0,n}$ is not. Clearly, $P_{1,n} = 0$ is a possible solution. Also note that the equations defining $P_{k,n}$ for even and odd k may be separated into otherwise identical equations. These lead to two inconsistent compatibility conditions for x_n unless one of these two subsequences of $P_{k,n}$ vanishes identically. In fact, $P_{l,n} = 0$ for odd l is the only possibility due to the inhomogeneity of the equation for $P_{0,n}$.

We are therefore left with the even polynomial

$$p_n = \sum_{k=0}^{m} P_{2k,n}\lambda^{2k}$$

where

$$P_{2m,n} = c_{m+2}$$
$$P_{2k-2,n} = (\Delta^2 + \Delta)^{-1}\mathcal{J}P_{2k,n} \qquad \text{for} \quad 1 \leqslant k \leqslant m$$

and $\mathcal{J}P_{0,n} = -2$.

Recursive substitution of the $P_{k,n}$ leads to the single compatibility condition

$$\mathcal{R}^m(x_{n+2} - x_n)c_{m+2} = -2 \qquad m \in \mathbb{N} \tag{2.7}$$

where $\mathcal{R} := J(\Delta^2 + \Delta)^{-1}$.

Below we explicitly list the solutions for different m up to $m = 3$.

- $m = 0$ yields the trivial linear nonautonomous equation, $\mathrm{d_0P_I}$:

$$x_n c_2 = c_1 + c_0(-1)^n - n$$

with

$$P_{0,n} = c_2 \neq 0.$$

- $m = 1$ yields the second-order equation, $\mathrm{d_2P_I}$:

$$x_n c_3(x_{n-1} + x_n + x_{n+1}) + x_n c_2 = c_1 + c_0(-1)^n - n \tag{2.8}$$

with

$$P_{0,n} = c_2 + c_3(x_n + x_{n-1})$$
$$P_{2,n} = c_3 \neq 0.$$

In this case, we have recovered the more general version of $\mathrm{dP_I}$ [17].

- $m = 2$ yields the fourth-order equation, $\mathrm{d_4P_I}$:

$$x_n c_4(x_{n+1}x_{n+2} + x_{n+1}^2 + 2x_n x_{n+1} + x_{n-1}x_{n-2} + x_{n-1}^2 + 2x_n x_{n-1} + x_n^2 + x_{n-1}x_{n+1})$$
$$+x_n c_3(x_{n-1} + x_n + x_{n+1}) + x_n c_2 = c_1 + c_0(-1)^n - n \tag{2.9}$$

with

$$P_{0,n} = c_2 + c_3(x_n + x_{n-1}) + c_4(2x_n x_{n-1} + x_{n-1}^2 + x_{n-1}x_{n-2} + x_n x_{n+1} + x_n^2)$$
$$P_{2,n} = c_3 + c_4(x_n + x_{n-1})$$
$$P_{4,n} = c_4 \neq 0.$$

- $m = 3$ yields the sixth-order equation, $\mathrm{d_6P_I}$:

$$x_n c_5(x_{n+1}x_{n+2}x_{n+3} + 2x_n x_{n+1}x_{n+2} + x_{n-1}x_{n+1}x_{n+2} + x_{n+1}^3 + x_{n+1}x_{n+2}^2 + 2x_{n+2}x_{n+1}^2$$
$$+3x_{n+1}x_n^2 + 3x_n x_{n+1}^2 + x_{n-1}x_{n-2}x_{n-3} + 2x_n x_{n-1}x_{n-2} + x_{n-2}x_{n-1}x_{n+1}$$
$$+x_{n-1}^3 + x_{n-2}^2 x_{n-1} + 2x_{n-1}^2 x_{n-2} + 3x_{n-1}x_n^2 + 3x_{n-1}^2 x_n + 4x_{n-1}x_n x_{n+1}$$
$$+x_n^3 + x_{n-1}x_{n+1}^2 + x_{n-1}^2 x_{n+1}) + x_n c_4(x_{n+1}x_{n+2} + x_{n+1}^2 + 2x_n x_{n+1}$$
$$+x_{n-1}x_{n-2} + x_{n-1}^2 + 2x_n x_{n-1} + x_n^2 + x_{n-1}x_{n+1})$$
$$+x_n c_3(x_{n-1} + x_n + x_{n+1}) + x_n c_2 = c_1 + c_0(-1)^n - n$$

with

$$
\begin{aligned}
P_{0,n} = {} & c_2 + c_3(x_n + x_{n-1}) + c_4(2x_nx_{n-1} + x_{n-1}^2 + x_{n-1}x_{n-2} + x_nx_{n+1} + x_n^2) \\
& + c_5(x_nx_{n+1}x_{n+2} + 2x_{n-1}x_nx_{n+1} + 3x_n^2x_{n-1} + 3x_{n-1}^2x_n + x_{n-2}^2x_{n-1} \\
& + 2x_{n+1}x_n^2 + x_n^3 + x_{n-1}^3 + x_nx_{n+1}^2 + x_{n-1}x_{n-2}x_{n-3} + 2x_nx_{n-1}x_{n-2} \\
& + 2x_{n-1}^2x_{n-2}) \\
P_{2,n} = {} & c_3 + c_4(x_n + x_{n-1}) + c_5(2x_nx_{n-1} + x_{n-1}^2 + x_{n-1}x_{n-2} + x_nx_{n+1} + x_n^2) \\
P_{4,n} = {} & c_4 + c_5(x_n + x_{n-1}) \\
P_{6,n} = {} & c_5 \neq 0.
\end{aligned}
$$

As equation (2.7) is solved for increasing m, the order of the compatibility condition increases by 2 at each stage.

2.2. Continuum limits

Continuum limits of difference equations found in section 2.1 are calculated.

2.2.1. The case $m = 1$, d_2P_I. Consider equation (2.8). If $c_0 = 0$, we recover the equation often referred to as $\mathrm{dP_I}$, with $\mathrm{P_I}$ as one of its continuum limits. If $c_0 \neq 0$ however, the presence of the $(-1)^n$ suggests an odd–even dependence in x_n. This dependence must be taken into account to obtain a meaningful continuum limit. This leads to the transformation

$$
x_{2k-1} = u_k \qquad x_{2k} = v_k.
$$

(This is similar to the limit pointed out by Grammaticos *et al* [3] for $\mathrm{dP_{II}}$ generalized with an additional $(-1)^n$ term.)

In our search for a continuum limit, we use the substitutions

$$
u_k = 1 + hy(kh) \qquad v_k = z(kh) \qquad t = kh \qquad c_3 = -\frac{2}{r_1}h^{-3}
$$

and for ease of notation rename

$$
\mu := \frac{c_1 + c_0}{c_3} \qquad \nu := \frac{c_1 - c_0 + 1}{c_3} \qquad \sigma := -\frac{c_2}{c_3} \qquad \text{and} \qquad \rho := -\frac{2}{c_3}.
$$

Then we find

$$
\begin{aligned}
z = {} & \tfrac{1}{2}(\sigma + \nu - 1) - \tfrac{1}{2}(\nu + 1)yh + \tfrac{1}{2}\left(r_1t + \nu y^2 - \tfrac{1}{2}\nu y_t - \tfrac{1}{2}y_t\right)h^2 \\
& + \tfrac{1}{2}\left(\tfrac{1}{2}r_1 + \nu yy_t - r_1ty - \nu y^3\right)h^3 + \mathrm{O}(h^4).
\end{aligned}
$$

Using the scalings

$$
\begin{aligned}
\mu &= -\tfrac{1}{4}\sigma^2 + \tfrac{1}{4}\nu^2 + \tfrac{1}{2}\nu + \sigma - \tfrac{3}{4} - r_2h^3 \\
\sigma &= \tfrac{3}{2} + \tfrac{1}{2}\nu^2 + r_3h^2 \\
\nu &= 1 + \tfrac{2}{3}r_4h
\end{aligned}
$$

in the limit $h \to 0$ we are left with

$$
y_{tt} - 2y^3 + 2r_4y^2 - 2r_1ty + 2r_3y + \tfrac{2}{3}r_1r_4t + r_1 + 2r_2 = 0.
$$

This is a scaled and translated version of $\mathrm{P_{II}}$.

2.2.2. The case $m = 2$, $d_4P_{\rm I}$. We first examine the case $c_0 = 0$ and illustrate the fact that second-order continuum limits are possible even though difference equation (2.9) is fourth order. Under the substitution

$$x_n = 1 + h^2u(nh) \qquad t = nh$$

and the scalings

$$10 + \frac{3c_3}{c_4} - \frac{c_1}{c_4} + \frac{c_2}{c_4} = r_1h^4 \qquad 20 + \frac{3c_3}{c_4} + \frac{c_1}{c_4} = r_2h^2 \qquad \frac{1}{c_4} = r_3h^5$$

equation (2.9) becomes

$$\left(10 + \frac{c_3}{c_4}\right)u_{tt} + \left(10 - \frac{c_1}{c_4}\right)u^2 + r_2u + r_1 + r_3t = 0$$

in the limit $h \to 0$. This is a scaled and translated version of $P_{\rm I}$. However, this case restricts the four degrees of freedom contained in the parameters of equation (2.9) to three.

Scalings that maintain the full four degrees of freedom, i.e.

$$10 + \frac{3c_3}{c_4} - \frac{c_1}{c_4} + \frac{c_2}{c_4} = r_1h^6 \qquad 20 + \frac{3c_3}{c_4} + \frac{c_1}{c_4} = r_2h^4$$
$$10 + \frac{c_3}{c_4} = r_3h^2 \qquad \frac{1}{c_4} = r_4h^7$$

lead, in the limit $h \to 0$, to the fourth-order equation

$$u_{tttt} + 5(u_t)^2 + 10uu_{tt} + r_3u_{tt} + 10u^3 + 3r_3u^2 + r_2u + r_1 + r_4t = 0. \tag{2.10}$$

This is a scaled and translated version of the fourth-order member of the $P_{\rm I}$ hierarchy [35, 36].

The case $c_0 \neq 0$ requires a new general approach to finding continuum limits in order to obtain a nontrivial limit.

Again, the presence of the $(-1)^n$ leads us to take

$$x_{2k-1} = U_k \qquad x_{2k} = V_k$$

whereupon equation (2.9) becomes a system in U and V. Assume $t = kh$ and expand each variable and parameter in this system as general Taylor series expansions in h. We obtain restrictions on the coefficients in these expansions by equating coefficients of successive powers of h. In demanding a nontrivial continuum limit (of order greater than 2) relationships between coefficients up to $O(h^5)$ are necessary. The continuum limit we obtain is

$$\begin{aligned} v_{tttt} + C_3v_{tt} - 5C_6(2vv_{tt} + (v_t)^2) - 10C_1(v(v_t)^2 + v^2v_{tt}) + 6C_1^2v^5 + C_2v^4 + C_4v^3 \\ +C_5v^2 + C_8v + C_7t(2C_1v + C_6) + C_9 = 0 \end{aligned} \tag{2.11}$$

where the C_i ($i = 1, \ldots, 9$) are constants and v is the $O(h)$ coefficient in the expansion of V_k. For details see the appendix. Equation (2.11) is a scaled and translated version of the fourth-order member of the $P_{\rm II}$ hierarchy (${}_4P_{\rm II}$) [29].

3. The $dP_{\rm II}$ hierarchy

We follow the procedure outlined in the previous section to find the $dP_{\rm II}$ hierarchy. We also give continuum limits of the first few members of this hierarchy.

3.1. The hierarchy

The linear problem associated with dP_{II} [3] is

$$\Phi_{n+1} = L_n(\lambda)\,\Phi_n \tag{3.1}$$

$$\frac{\partial \Phi_n}{\partial \lambda} = M_n(\lambda)\,\Phi_n \tag{3.2}$$

where

$$L = \begin{pmatrix} \lambda & x_n \\ x_n & 1/\lambda \end{pmatrix} \qquad M = \begin{pmatrix} A_n & B_n \\ C_n & -A_n \end{pmatrix}$$

and A, B, C are rational in λ.

The compatibility of equations (3.1) and (3.2) gives three equations defining A, B and C. They reduce to a single equation in A by using two of these equations to eliminate B and C in terms of A [3],

$$\begin{aligned}\lambda^2 x_{n+1}x_{n-1}(A_n - A_{n+1}) + \lambda x_{n+1}x_{n-1} + x_{n+1}x_n(A_n(1+x_{n-1}^2) + A_{n-1}(x_{n-1}^2 - 1))\\ -x_n x_{n-1}(A_{n+1}(1+x_{n+1}^2) + A_{n+2}(x_{n+1}^2 - 1))\\ -\frac{1}{\lambda}x_n(x_{n+1}+x_{n-1}) - \frac{1}{\lambda^2}x_{n+1}x_{n-1}(A_{n+1} - A_n) + \frac{1}{\lambda^3}x_{n+1}x_{n-1} = 0.\end{aligned}$$

This equation may be rewritten in operator form (using $\Delta A_n := A_{n+1} - A_n$)

$$\lambda + \frac{1}{\lambda^3} - \frac{x_{n+1}}{\lambda}\left(\frac{1}{x_{n+2}} + \frac{1}{x_n}\right) = \left[\left(\lambda^2 + \frac{1}{\lambda^2}\right)(\Delta^2 + \Delta) + \mathcal{J}\right] A_n \tag{3.3}$$

where

$$\mathcal{J} := x_{n+1}\left[\Delta(\Delta+2)\left(\left(x_n - \frac{1}{x_n}\right)\Delta + 2x_n\right) - \frac{2}{x_n}\Delta\right].$$

Equation (3.3) suggests that A_n is rational in λ. In this case it turns out that only odd degree terms in λ and $1/\lambda$ are needed in the expansion of A_n. Without loss of generality we consider

$$A_n := a_{1,n}\frac{1}{\lambda} + \sum_{k=1}^{m}\left(\lambda^{2k-1} + \frac{1}{\lambda^{2k+1}}\right) a_{2k+1,n}.$$

Substituting this expansion in (3.3) and solving for the coefficients a_{2i+1} $(i = 0, \ldots, m)$ we obtain

$$\begin{aligned}&a_{2m+1,n} = c_{m+2} \qquad a_{2m-1,n} = -2c_{m+2}x_n x_{n-1} + c_{m+1}\\ &a_{2k-3,n} = -a_{2k+1,n} - (\Delta^2+\Delta)^{-1}\mathcal{J}a_{2k-1,n} \qquad \text{for} \quad k = 3, \ldots, m\end{aligned}$$

and

$$a_{1,n} = n - a_{5,n} - (\Delta^2+\Delta)^{-1}\mathcal{J}a_{3,n}$$

with the compatibility condition

$$x_{n+1}\left(\frac{1}{x_{n+2}} + \frac{1}{x_n}\right) = -2(\Delta^2+\Delta)a_{3,n} - \mathcal{J}a_{1,n}. \tag{3.4}$$

Equation (3.4) may be rewritten after recursive substitution of the a_{2i+1} as

$$x_{n+1}\left(\frac{1}{x_{n+2}} + \frac{1}{x_n}\right) = \mathcal{L}_{m-2}(c_{m+2}) - \mathcal{L}_{m-1}(c_{m+1} - 2c_{m+2}x_n x_{n-1}) - \mathcal{J}n$$

for $m > 0$, where

$$\mathcal{L}_i := -\mathcal{L}_{i-2} - \mathcal{L}_{i-1}(\Delta^2 + \Delta)^{-1}\mathcal{J} \qquad \text{for} \quad i = 3, \ldots, m-1$$
$$\mathcal{L}_0 := \mathcal{J} \qquad \mathcal{L}_1 := 2(\Delta^2 + \Delta) - \mathcal{R}\mathcal{J} \qquad \mathcal{L}_2 := -3\mathcal{J} + \mathcal{R}^2\mathcal{J}$$

and

$$\mathcal{R} := \mathcal{J}(\Delta^2 + \Delta)^{-1}.$$

We list the first three equations in the hierarchy. (The fourth equation ($m = 3$) is a sixth-order equation which is too long to list here.)

- $m = 0$ yields the trivial linear nonautonomous equation, $\mathrm{d}_0\mathrm{P}_{\mathrm{II}}$:

$$x_n(2c_2 + 2n + 1) = c_1 + c_0(-1)^n$$

with

$$a_{1,n} = c_2 + n.$$

- $m = 1$ yields the second-order equation, $\mathrm{d}_2\mathrm{P}_{\mathrm{II}}$:

$$2c_3(x_{n+1} + x_{n-1})(1 - x_n^2) + x_n(2c_2 + 2n + 1) = c_1 + c_0(-1)^n \tag{3.5}$$

with

$$a_{1,n} = c_2 - 2c_3x_nx_{n-1} + n$$
$$a_{3,n} = c_3 \neq 0.$$

Equation (3.5) is the general form of $\mathrm{dP}_{\mathrm{II}}$ [3] traditionally written as

$$x_{n+1} + x_{n-1} = \frac{(C_1 + C_2n)x_n + C_3 + C_4(-1)^n}{1 - x_n^2} \tag{3.6}$$

where $C_1 := -(2c_2 + 1)/(2c_3)$, $C_2 := -1/c_3$, $C_3 := c_1/(2c_3)$ and $C_4 := c_0/(2c_3)$.

- $m = 2$ yields the fourth-order equation, $\mathrm{d}_4\mathrm{P}_{\mathrm{II}}$:

$$2c_4(x_{n+2}(1 - x_{n+1}^2) + x_{n-2}(1 - x_{n-1}^2))(1 - x_n^2) - 2c_4x_n(x_{n+1} + x_{n-1})^2(1 - x_n^2)$$
$$+2c_3(x_{n+1} + x_{n-1})(1 - x_n^2) + x_n(2c_2 + 2n + 1) = c_1 + c_0(-1)^n \tag{3.7}$$

with

$$a_{1,n} = c_2 - 2c_3x_nx_{n-1} + n - 2c_4(x_nx_{n-2} + x_{n+1}x_{n-1})$$
$$+2c_4x_nx_{n-1}(x_{n-1}x_{n-2} + x_{n+1}x_n + x_{n-1}x_n)$$
$$a_{3,n} = c_3 - 2c_4x_nx_{n-1}$$
$$a_{5,n} = c_4 \neq 0.$$

We write (3.7) as

$$x_{n+2}(1 - x_{n+1}^2) + x_{n-2}(1 - x_{n-1}^2) = (x_{n+1} + x_{n-1})(x_n(x_{n+1} + x_{n-1}) + C_5)$$
$$+\frac{(C_1 + C_2n)x_n + C_3 + C_4(-1)^n}{1 - x_n^2} \tag{3.8}$$

where $C_1 := -(2c_2 + 1)/(2c_4)$, $C_2 := -1/c_4$, $C_3 := c_1/(2c_4)$, $C_4 := c_0/(2c_4)$, and $C_5 := -c_3/c_4$.

3.2. Continuum limits

3.2.1. The case $m = 1$, d_2P_{II}. When $C_4 = 0$, equation (3.6) is $\mathrm{dP}_{\mathrm{II}}$ with P_{II} as one of its continuum limits. When $C_4 \neq 0$, equation (3.6) has $\mathrm{P}_{\mathrm{III}}$ as a continuum limit [3].

3.2.2. The case $m = 2$, d_4P_{II}. Firstly, for the case $C_4 = 0$ we see that under the substitution

$$x_n = hu(nh) \qquad t = nh$$

scalings that give P_{II} as a second-order continuum limit may be found, but the scalings

$$C_1 = -6 - 2h^2r_1 - 2h^3r_2 \qquad C_2 = h^5r_3 \qquad C_3 = h^5r_4$$
$$C_5 = 4 + h^2r_1 + h^3r_2$$

lead, in the limit $h \to 0$, to the fourth-order equation

$$u_{tttt} - 10u^2u_{tt} - 10u(u_t)^2 + 2u^3r_1 + 6u^5 - ur_3t - r_1u_{tt} - r_4 = 0. \tag{3.9}$$

Equation (3.9) is a scaled and translated version of ${}_4P_{II}$ [29].

For the case $C_4 \neq 0$ we use a similar approach to that used to find the continuum limit of d_4P_I, $c_0 \neq 0$ (subsection 2.2.2).

The $(-1)^n$ term in equation (3.8) prompts us to take $x_{2k-1} = U_k$, $x_{2k} = V_k$. In the resulting system, we find that general expansions

$$t = kh$$
$$U_k = \frac{\mathrm{i}}{hv_1(t)} + u_2(t) + hu_3(t) + h^2u_4(t) + h^3u_5(t) + \mathrm{O}(h^4)$$
$$V_k = \frac{\mathrm{i}v_1(t)}{h} + v_2(t) + hv_3(t) + h^2v_4(t) + h^3v_5(t) + \mathrm{O}(h^4)$$
$$C_1 = \frac{2}{h^4} + \frac{2r_1}{h^3} \qquad C_2 = \frac{r_2}{h} \qquad C_3 = \frac{\mathrm{i}r_3}{h}$$
$$C_4 = \frac{\mathrm{i}r_4}{h} \qquad C_5 = \frac{2}{h^2} + \frac{r_1}{h}$$

with added relationships for the $u_j(t)$, $j = 2, 3, 4, 5$, in terms of $v_l(t)$, $1 \leqslant l \leqslant j$, make order h terms vanish until $\mathrm{O}(h)$ where in the limit $h \to 0$ we have the fourth-order equation

$$v_1'''' + 10v_1^2v_1'' - \frac{v_1''}{2}(4r_2t - r_1^2) - 2r_2v_1' - \frac{4v_1'v_1'''}{v_1} + \frac{21v_1''(v_1')^2}{2v_1^2} - \frac{3(v_1'')^2}{v_1} + \frac{10v_1''}{v_1^2} - \frac{9(v_1')^4}{2v_1^3}$$
$$- \frac{20(v_1')^2}{v_1^3} + \frac{(v_1')^2}{2v_1}(4r_2t - r_1^2) + 8v_1^5 - 2v_1^3(4r_2t - r_1^2) + 8v_1^2(r_3 - r_4) - \frac{8}{v_1^3}$$
$$+ \frac{2}{v_1}(4r_2t - r_1^2) - 8(r_3 + r_4) = 0.$$

We propose that the above equation is, in fact, the fourth-order equation in the P_{III} hierarchy.

4. The dP_{XXXIV} hierarchy

Recently Joshi *et al* [26] have proposed an algorithmic method for deriving Miura transformations for discrete equations. It has been used to construct Miura transformations for a number of examples and, in particular, that linking dP_{II} with dP_{XXXIV}.

In this section, we use the same method to find a Miura transformation associated with d_4P_{II} and hence find a fourth-order equation that we propose is the next equation in the dP_{XXXIV} hierarchy. A similar approach may be applied to higher-order equations in the dP_{II} hierarchy thereby providing a method by which to construct the full dP_{XXXIV} hierarchy.

The starting point of this algorithm is the associated τ functions. For $\mathrm{dP_{II}}$ (equation (1.2)) these are given by F_n and G_n where the bilinearizing transformation

$$x_n = 1 - \frac{F_{n+1}G_{n-1}}{F_nG_n} = -1 + \frac{F_{n-1}G_{n+1}}{F_nG_n}$$

may be found by considering the singularity structure $\{\pm 1, \infty, \mp 1\}$ [22].

Now rewrite these in terms of discrete logarithmic derivatives

$$x_n = 1 - \frac{u_n}{v_n} = -1 + \frac{u_{n-1}}{v_{n+1}} \tag{4.1}$$

where $u_n = F_n/F_{n+1}$ and $v_n = G_{n-1}/G_n$. Elimination of one of the u, v by combining (4.1) and (1.2), leads to a transformed equation, in this case $\mathrm{dP_{XXXIV}}$

$$(w_{n+1} + w_n - z_{n+1})(w_n + w_{n-1} - z_n) = \frac{(2w_n - C_3 - z_n)(2w_n + C_3 - z_{n+1})}{w_n} \tag{4.2}$$

where $z_n = C_1 + C_2 n$. Equation (4.2) is linked to $\mathrm{dP_{II}}$ by the Miura transformation

$$x_n = \frac{w_n - w_{n-1} - C_3}{z_n}.$$

We now consider a similar construction for $\mathrm{d_4P_{II}}$. First, we find that the singularity structure for $\mathrm{d_4P_{II}}$ is $\{\pm 1, \infty, \mp 1\}$. Thus we consider associated τ functions, bilinearizing transformations and logarithmic derivative versions thereof to be the same as for $\mathrm{dP_{II}}$. Then by eliminating one of the u, v by combining (4.1) and (3.8), we find the transformed equation

$$\begin{aligned}
w_n^2(w_{n+2}w_{n+1} + w_{n-1}w_{n-2})(w_n - \kappa) + w_n^5 - 2w_n^4(\kappa + 2)& \\
-w_n(z_n w_{n+2}w_{n+1} + z_{n+1}w_{n-1}w_{n-2})& \\
+w_{n+1}w_nw_{n-1}(w_{n+2} + w_{n+1} + w_{n-1} + w_{n-2})(2w_n - \kappa)& \\
+w_n^2(w_{n+2}w_{n+1}^2 + w_{n-1}^2 w_{n-2})& \\
+w_{n+1}w_nw_{n-1}(w_{n+2} + w_{n+1})(w_{n-1} + w_{n-2})& \\
+w_n^2(w_{n+1}^2 + w_{n-1}^2)(3w_n - 2\kappa - 4) - w_n(z_n w_{n+1}^2 + z_{n+1}w_{n-1}^2)& \\
+w_{n+1}w_nw_{n-1}(\kappa^2 - 4w_n\kappa - 8w_n + 5w_n^2) + w_n^2(w_{n+1}^3 + w_{n-1}^3)& \\
+w_n^2(w_{n+1} + w_{n-1})(\kappa^2 + 8\kappa + 3w_n^2 - 4w_n\kappa - 8w_n)& \\
-w_n^2((2z_n + z_{n+1})w_{n+1} + (2z_{n+1} + z_n)w_{n-1})& \\
+w_n((2z_{n+1} + z_n(\kappa + 2))w_{n+1} + (2z_n + z_{n+1}(\kappa + 2))w_{n-1})& \\
+w_n^3(-(z_n + z_{n+1}) + \kappa(\kappa + 8)) + w_n^2((z_n + z_{n+1})(\kappa + 2) - 4\kappa^2)& \\
+w_n(z_nz_{n+1} - 2\kappa(z_n + z_{n+1})) + (C_3 + z_n)(C_3 - z_{n+1}) = 0&
\end{aligned} \tag{4.3}$$

where $\kappa = C_5 + 2$. Equation (4.3) is linked to $\mathrm{d_4P_{II}}$ by the Miura transformation

$$x_n = -\frac{(w_n - w_{n-1})(w_n + w_{n-1} - \kappa) + w_{n+1}w_n - w_{n-1}w_{n-2} + C_3}{(w_n + w_{n-1})(w_n + w_{n-1} - \kappa) + w_{n+1}w_n + w_{n-1}w_{n-2} + z_n}.$$

We propose that equation (4.3) is the fourth-order equation in the $\mathrm{dP_{XXXIV}}$ hierarchy.

5. Discussion

In this paper we have presented the discrete $\mathrm{P_I}$ and $\mathrm{P_{II}}$ hierarchies in operator form. We have shown their connection with hierarchies of ODEs through continuum limit calculations. The connection of the $\mathrm{dP_{II}}$ hierarchy with the $\mathrm{dP_{XXXIV}}$ hierarchy is also shown through Miura transformations.

These results reveal a rich underlying structure that is open to further investigation. For example, in deriving discrete Painlevé hierarchies in operator form through transformations of already existing operator generated discrete Painlevé hierarchies. An obvious starting point would be with the $\mathrm{dP}_{\mathrm{XXXIV}}$ hierarchy as a Miura transformation of the $\mathrm{dP}_{\mathrm{II}}$ hierarchy. On a larger scale, a study of the properties of the difference operators associated with each hierarchy would provide insight into what governs the hierarchies and would enable access to important information such as special solutions or special integrals.

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Appendix

Details of the continuum limit calculation of $\mathrm{d}_4\mathrm{P}_{\mathrm{I}}$, $c_0 \neq 0$ (subsection 2.2.2) are presented. Consider equation (2.9). Let

$$x_{2k-1} = U_k \qquad x_{2k} = V_k$$

and rename

$$\mu := \frac{c_3}{c_4} \qquad \tau := \frac{c_2}{c_4}$$
$$\beta := \frac{c_1 + c_0}{c_4} \qquad \omega := \frac{c_1 - c_0 + 1}{c_4} \qquad \text{and} \qquad \rho := -\frac{2}{c_4}.$$

Equation (2.9) now becomes the system

$$U_{k+1}V_k + V_k^2 + 2V_kU_k + U_{k-1}V_{k-1} + V_{k-1}^2 + 2U_kV_{k-1} + U_k^2 + V_{k-1}V_k$$
$$+\mu(U_k + V_k + V_{k-1}) + \tau - \frac{\omega + \rho k}{U_k} = 0 \tag{A.1}$$
$$U_{k+1}V_{k+1} + U_{k+1}^2 + 2V_kU_{k+1} + U_kV_{k-1} + U_k^2 + 2U_kV_k + V_k^2 + U_kU_{k+1}$$
$$+\mu(U_{k+1} + U_k + V_k) + \tau - \frac{\beta + \rho k}{V_k} = 0. \tag{A.2}$$

Assume the following general Taylor series expansions:

$$t = kh \qquad \rho = r_1h^5 + \mathrm{O}(h^6)$$
$$U_k = u_0(t) + hu_1(t) + h^2u_2(t) + h^3u_3(t) + h^4u_4(t) + h^5u_5(t) + \mathrm{O}(h^6)$$
$$V_k = v_0(t) + hv_1(t) + h^2v_2(t) + h^3v_3(t) + h^4v_4(t) + h^5v_5(t) + \mathrm{O}(h^6)$$
$$\mu = \mu_0 + h\mu_1 + h^2\mu_2 + h^3\mu_3 + h^4\mu_4 + h^5\mu_5 + \mathrm{O}(h^6)$$
$$\omega = \omega_0 + h\omega_1 + h^2\omega_2 + h^3\omega_3 + h^4\omega_4 + h^5\omega_5 + \mathrm{O}(h^6)$$
$$\tau = \tau_0 + h\tau_1 + h^2\tau_2 + h^3\tau_3 + h^4\tau_4 + h^5\tau_5 + \mathrm{O}(h^6)$$
$$\beta = \beta_0 + h\beta_1 + h^2\beta_2 + h^3\beta_3 + h^4\beta_4 + h^5\beta_5 + \mathrm{O}(h^6).$$

Relationships that give a nontrivial continuum limit (of order greater than 2) of the system (A.1)–(A.2) are

$$u_0(t) = v_0(t) = v_0 = \text{constant} \qquad \mu_0 = -2v_0$$
$$\beta_0 = \omega_0 = 2v_0^3 \qquad \beta_1 = \omega_1 \qquad \beta_2 = \omega_2 \qquad \beta_3 = \omega_3 \qquad \beta_4 = \omega_4$$

$$\tau_0 = -2v_0^2 \qquad \tau_1 = 2\mu_1 v_0 \qquad \tau_2 = -\mu_2 v_0 - \frac{\omega_2}{v_0} + \mu_1^2$$

$$\tau_3 = -\mu_3 v_0 - \frac{\omega_3}{v_0} + \frac{1}{2}\mu_2\mu_1 + \frac{\mu_1^3}{2v_0} - \frac{\omega_2\mu_1}{2v_0^2}$$

and

$$u_1 + v_1 = -\mu_1$$

$$u_2 + v_2 = \frac{1}{2}v_1' + \frac{1}{2v_0}v_1^2 + \frac{\mu_1}{2v_0}v_1 - \frac{\mu_2}{4} + \frac{\omega_2}{4v_0^2} - \frac{\mu_1^2}{4v_0}$$

$$u_3 + v_3 = -\frac{1}{4}(u_2' - v_2') + \frac{1}{4v_0}(2v_1 + 3\mu_1)v_2 + \frac{1}{4v_0}(-2v_1 + \mu_1)u_2 \\ -\frac{\mu_1}{4v_0}v_1' - \frac{\mu_3}{4} + \frac{\omega_3}{4v_0^2} + \frac{\omega_2\mu_1}{8v_0^3} - \frac{\mu_1^3}{8v_0^2} + \frac{\mu_1\mu_2}{8v_0}$$

$$u_4 + v_4 = \frac{1}{4}(v_3 - u_3) + u_3\left(\frac{\mu_1}{4v_0} - \frac{v_1}{2v_0}\right) + v_3\left(\frac{3\mu_1}{4v_0} + \frac{v_1}{2v_0}\right) \\ -\frac{3}{16}(u_2'' + v_2'') + \frac{\mu_1}{8v_0}(u_2' - v_2') - \frac{1}{8v_0}(u_2^2 + v_2^2) \\ +(u_2 + v_2)\left(\frac{3}{8v_0}v_1' - \frac{3}{8v_0^2}v_1^2 - \frac{3\mu_2}{16v_0} - \frac{\omega_2}{16v_0^3} - \frac{3\mu_1}{8v_0^2}v_1\right) \\ -\frac{3}{4v_0}u_2v_2 + v_1'\left(\frac{\mu_2}{8v_0}\right) + v_1''\left(\frac{\mu_1}{16v_0} + \frac{v_1}{8v_0}\right) + \frac{1}{12}v_1''' \\ +v_1^4\left(\frac{1}{4v_0^3}\right) + v_1^3\left(\frac{\mu_1}{2v_0^3}\right) + v_1^2\left(\frac{3\mu_1^2}{16v_0^3} + \frac{\omega_2}{8v_0^4}\right) + v_1\left(-\frac{\mu_1^3}{16v_0^3} + \frac{\omega_2\mu_1}{8v_0^4}\right) \\ +\frac{a}{8v_0^2}t - \frac{3\mu_4}{8} + \frac{3\mu_3\mu_1}{16v_0} + \frac{\omega_4}{8v_0^2} - \frac{\mu_1^4}{16v_0^3} + \frac{\omega_3\mu_1}{16v_0^3} + \frac{\omega_2\mu_1^2}{16v_0^4} - \frac{\tau_4}{8v_0}$$

where $'$ denotes $\frac{\mathrm{d}}{\mathrm{d}t}$. Under these conditions the coefficients of O(1) to O(h^4) vanish and we are left with an equation at O(h^5),

$$v_{tttt} + C_3 v_{tt} - 5C_6(2vv_{tt} + (v_t)^2) - 10C_1(v(v_t)^2 + v^2 v_{tt}) + 6C_1^2 v^5 \\ +C_2v^4 + C_4v^3 + C_5v^2 + C_8v + C_7t(2C_1v + C_6) + C_9 = 0$$

where

$$C_1 := \frac{1}{v_0^2} \qquad C_2 := 15C_1C_6 \qquad C_3 := -\frac{1}{v_0}\left(3\mu_2 + \frac{\mu_1^2}{v_0} + \frac{\omega_2}{v_0^2}\right)$$

$$C_4 := 10C_6^2 - 2C_1C_3 \qquad C_5 := -3C_3C_6 \qquad C_6 := \frac{\mu_1}{v_0^2} \qquad C_7 := \frac{4a}{v_0}$$

$$C_8 := \frac{5\omega_2\mu_1^2}{v_0^5} + \frac{4\omega_3\mu_1}{v_0^4} + \frac{3\mu_1^2\mu_2}{v_0^3} + \frac{8\mu_4}{v_0} + \frac{8\omega_4}{v_0^3} - \frac{7\mu_1^4}{2v_0^4} + \frac{\mu_2\omega_2}{v_0^4} \\ -\frac{\mu_2^2}{2v_0^2} + \frac{8\tau_4}{v_0^2} - \frac{4\mu_3\mu_1}{v_0^2} - \frac{\omega_2^2}{2v_0^6}$$

$$C_9 := \frac{4\mu_4\mu_1}{v_0} + \frac{2\omega_2\mu_1^3}{v_0^5} - \frac{7\mu_1^5}{4v_0^4} + \frac{8\omega_5}{v_0^2} - \frac{8\beta_5}{v_0^2} - \frac{\mu_1\mu_2^2}{4v_0^2} + \frac{\mu_1\mu_2\omega_2}{2v_0^4} \\ +\frac{4\omega_4\mu_1}{v_0^3} - \frac{\mu_1\omega_2^2}{4v_0^6} - \frac{2\mu_3\mu_1^2}{v_0^2} + \frac{4\mu_1\tau_4}{v_0^2} + \frac{2\omega_3\mu_1^2}{v_0^4} + \frac{4a}{v_0^2}$$

and $v_1(t) = v(t)$ for conciseness. We could choose special values for the constants v_0, μ_i, ω_i, τ_i and β_i ($i = 0, 1, \ldots, 5$) but we wish to illustrate the strength of this approach where little ingenuity is required.

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